THE CRITICAL SLAB PROBLEM FOR REFLECTING BOUNDARY CONDITIONS WITH THE H_N METHOD

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Introduction

The angular density, for the neutrons which are introduced into the system only through fission with no extraneous sources obeys the homogeneous time independent, one velocity, linear transport equation $[1-3]$.

$$\vec{\Omega}.\vec{\nabla}\psi(\vec{r},\vec{\Omega})+\psi(\vec{r},\vec{\Omega})=\frac{c}{4\pi}\int f(\vec{\Omega}'\to\vec{\Omega})\psi(\vec{r},\vec{\Omega})d\vec{\Omega}' \tag{1}$$

where, $\psi(\vec{r},\vec{\Omega})$ is the angular density or the single particle distribution function giving the average number of particles within unit volume around the space point $\vec{r}$ and moving within unit solid angle along the direction $\vec{\Omega}$, $f(\vec{\Omega}'\to\vec{\Omega})$ is the probability distribution function for a particle after collision moving in the direction $\vec{\Omega}$, assuming that it was moving in the direction $\vec{\Omega}'$ before collision. c is the average number of secondaries produced by collision and distances are measured in mean free paths.
For isotropic scattering in plane geometry Eq.(1) becomes

$$\mu\frac{\partial\psi(x,\mu)}{\partial x}+\psi(x,\mu)=\frac{c}{2}\int_{-1}^{+1}\psi(x,\mu')d\mu'. \tag{2}$$

The solution of Eq.(1) is given by the normal mode expansion as

$$\begin{aligned}\psi(x,\mu)&=A(\nu_0)\phi(\nu_0,\mu)e^{-\frac{x}{\nu_0}}+A(-\nu_0)\phi(-\nu_0,\mu)e^{\frac{x}{\nu_0}}\\&+\int_0^1 A(\nu)\phi(\nu,\mu)e^{-\frac{x}{\nu}}d\nu+\int_0^1 A(-\nu)\phi(-\nu,\mu)e^{\frac{x}{\nu}}d\nu.\end{aligned} \tag{3}$$

$A(\pm\nu_0)$ and $A(\pm\nu)$ represent the expansion coefficients, $\phi(\pm\nu_0,\mu)$ and $\phi(\pm\nu,\mu)$ are the discrete and continuous eigenfunctions corresponding to discrete and continuous eigenvalues $\pm\nu_0$ and $\pm\nu$ respectively.

$$\phi(\pm\nu_0,\mu)=\frac{c\nu_0}{2}\frac{1}{\nu_0\mp\mu} \tag{4}$$

$$\phi(\nu,\mu)=\frac{c\nu}{2}P\frac{1}{\nu-\mu}+\lambda(\nu)\delta(\nu-\mu) \tag{5}$$

if ν does not lie on the real line between -1 and 1 then

$$\Lambda(\nu)=1-c\nu\tanh^{-1}\frac{1}{\nu}=0 \tag{6}$$

leads to discrete eigenvalues. P denotes the Cauchy principle value.

The Critical Slab Problem

Consider an homogeneous slab of half thickness a which is centered at the origin $[1,2,3]$.
We have the symmetry condition

$$\psi(x,\mu)=\psi(-x,-\mu). \tag{7}$$

Eq.(3) can be written as

$$\psi(x,\mu)=A(v_0)\left[\phi(v_0,\mu)e^{-\frac{x}{v_0}}+\phi(-v_0,\mu)e^{\frac{x}{v_0}}\right]$$

$$+\int_0^1 A(v)\left[\phi(v,\mu)e^{-\frac{x}{v}}+\phi(-v,\mu)e^{\frac{x}{v}}\right]dv. \tag{8}$$

The slab which is surrounded by the same reflecting medium from both sides can be represented with the following boundary conditions,

$$\psi(a,-\mu)=R\psi(a,\mu),\quad for\,\mu>0$$
$$\psi(-a,\mu)=R\psi(-a,-\mu).\quad for\,\mu>0 \tag{9}$$

If $R=0$ then boundary conditions reduce to the boundary conditions of a slab surrounded by a vacuum.

In the criticality problem we consider $c>1$; that is, a reactor can not be critical if $c<1$. These kinds of solutions will exist only when there is a definite relation between c and the size of the slab. This relation, is called the criticality condition. Then the problem is to obtain this condition with the reflecting boundary conditions Eq.(9).

The Use of the H_N method

To obtain the criticality condition and therefore the variation of the critical slab thickness with the number of secondaries c we consider the angular flux at the boundary of the slab. The angular flux at $x=a$ is

$$\psi(a,\mu)=A(v_0)\left[\phi(v_0,\mu)e^{-\frac{a}{v_0}}+\phi(-v_0,\mu)e^{\frac{a}{v_0}}\right]$$

$$+\int_0^1 A(v)\left[\phi(-v,\mu)e^{\frac{a}{v}}+\phi(v,\mu)e^{-\frac{a}{v}}\right]dv.\quad -1\le\mu\le 1 \tag{10}$$

Calculation of the Coefficients $A(v_0)$ and $A(v)$

Multiplying Eq.(10) by $\mu\phi(v_0,\mu)$ and $\mu\phi(v,\mu)$ respectively and integrating over $\mu\in(-1,1)$ and using the following full range orthogonality conditions

$$\int_{-1}^{1}\mu\phi^2(\pm v_0,\mu)d\mu=N(\pm v_0), \tag{11}$$

$$\int_{-1}^{1}\mu\phi(v,\mu)\phi(v',\mu)d\mu=N(v)\delta(v-v'), \tag{12}$$

$$N(\mu v_0)=\mu\frac{cv_0^3}{2}\left[\frac{c}{v_0^2-1}-\frac{1}{v_0^2}\right], \tag{13}$$

$$N(v)=v\left[\left(1-cv\tanh^{-1}v\right)^2+\left(\frac{\pi N}{2}\right)^2\right], \tag{14}$$

and defining the following expansions

$$\psi(a,\propto)=\psi(-a,-\propto)=\sum_{\alpha=0}^{N}a_{\alpha}\propto^{\alpha}\ ,\ \propto>0 \qquad (15)$$

we obtain

$$A(\zeta)=\frac{c\zeta}{2}\frac{e^{-a/\zeta}}{N(\zeta)}\sum_{\alpha=0}^{N}a_{\alpha}\left[-A_{\alpha}(\zeta)+RB_{\alpha}(\zeta)\right] \qquad (16)$$

where

$$A_l(\zeta)=\frac{2}{c\zeta}\int_0^1\propto^{l+1}\phi(-\zeta,\propto)\ d\propto,\ B_l(\zeta)=\frac{2}{c\zeta}\int_0^1\propto^{l+1}\phi(-\zeta,\propto)\ d\propto, \qquad (17)$$

$$A_0(\zeta)=1-\zeta\ln\left(1+\frac{1}{\zeta}\right)\ ,\ B_0(\zeta)=\frac{2}{c}-1-\zeta\ln\left(1+\frac{1}{\zeta}\right), \qquad (18)$$

$$A_l(\zeta)=\frac{1}{l+1}-\zeta A_{l-1}(\zeta),\ B_l(\zeta)=\zeta B_{l-1}(\zeta)-\frac{1}{l+1}, \qquad (19)$$

$$\zeta=\nu_0 \text{ or } \nu \qquad (20)$$

Now we consider the entering angular flux at $x=a$ with

$$\psi(a,-\propto)=A(\nu_0)\left[\phi(\nu_0,-\propto)e^{-\frac{a}{\nu_0}}+\phi(-\nu_0,-\propto)e^{\frac{a}{\nu_0}}\right]$$

$$+\int_0^1 A(\nu)\left[\phi(-\nu,-\propto)e^{\frac{a}{\nu}}+\phi(\nu,-\propto)e^{-\frac{a}{\nu}}\right]d\nu,\quad \propto>0 \qquad (21)$$

similarly, one can write the entering angular flux $\psi(-a,\propto)$ at $x=-a$, for $\propto>0$.

To get the criticality condition we multiply Eq.(21) with $\propto^{m+1}$ and integrate over $\propto\in(0,1)$, $m=0,1,2,\ldots N$.

The resulting equation together with Eqs.(15), (16) lead to

$$\begin{aligned}
&\sum_{\alpha}^{N}a_{\alpha}\left\{\left(\frac{c\nu_0}{2}\right)^2\left[\frac{A_{\alpha}(\nu_0)A_m(\nu_0)}{N(\upsilon_0)}e^{-\frac{2a}{\nu_0}}+\frac{A_{\alpha}(\nu_0)A_m(\nu_0)}{N(\nu_0)}\right]\right.\\
&+\left(\frac{c}{2}\right)^2\left[\int_0^1\frac{\nu^2}{N(\nu)}A_{\alpha}(\nu)A_m(\nu)e^{-\frac{2a}{\nu}}d\nu+\int_0^1\frac{\nu^2}{N(\nu)}A_{\alpha}(\nu)A_m(\nu)d\nu\right]\\
&=R\left\{-\frac{1}{\alpha+m+2}+\left[\frac{B_{\alpha}(\nu_0)A_m(\nu_0)}{N(\nu_0)}e^{-\frac{2a}{\nu_0}}+\frac{B_{\alpha}(\nu_0)B_m(\nu_0)}{N(\nu_0)}\right]\right.\\
&\left.\left.+\left(\frac{c}{2}\right)^2\left[\int_0^1\frac{\nu^2}{N(\nu)}B_{\alpha}(\nu)A_m(\nu)e^{-\frac{2a}{\nu}}d\nu+\int_0^1\frac{\nu^2}{N(\nu)}B_{\alpha}(\nu)B_m(\nu)d\nu\right]\right\}\right\}.
\end{aligned} \qquad (22)$$

To calculate the critical slab thickness, since $c>1$, ν_0 should be replaced by ν_0/i.

Conclusions

H_N method [4] which we have used to solve the critical slab problem for reflecting boundary conditions leads to the exact criticality condition given in Eq.(22). Using this equation we have calculated the numerical values of the critical half thicknesses corresponding to different values of the reflection coefficients R for $c = 1.1$ and 1.2.

The results are tabulated in Tables (1) and (2) together with the numerical results given by the Case's method [5]. It is shown that the H_N method provides fast converging numerical results.

Finally we may conclude that the method is very effective because of leading to an exact criticality equation and because of lower order approximations need to be considered.

***Table 1**: The slab critical thickness for c=1.1*

N	R=0	R=0.25	R=0.5	R=0.75	R=0.99
0	4.22483	3.54384	2.56541	1.31663	0.06313
1	4.22661	3.54014	2.56659	1.30847	0.04861
2	4.22662	3.54066	2.56652	1.30745	0.04867
3	4.22662	3.54063	2.56651	1.30749	0.04866
4	4.22662	3.54060	2.56680	1.30765	0.04868
Atalay	4.22653	3.54352	2.27212	1.31222	0.04887

***Table 2**: The slab critical thickness for c=1.5*

N	R=0	R=0.25	R=0.5	R=0.75	R=0.99
0	1.21009	0.87649	0.55295	0.26493	0.01185
1	1.21012	0.87600	0.54430	0.24612	0.00876
2	1.21011	0.87594	0.54423	0.24617	0.00878
3	1.21011	0.87603	0.54416	0.24617	0.00878
4	1.21011	0.87604	0.54417	0.24616	0.00878
Atalay	1.20928	0.86761	0.54481	0.24650	0.00880

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