THE EFFECT OF EXTREMELY ANISOTROPIC SCATTERING ON THE CRITICAL SLAB PROBLEM IN ONE-SPEED NEUTRON TRANSPORT THEORY

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ABSTRACT

To solve the critical slab problem, the infinite medium Green's function for the anisotropic scattering obtained in terms of the infinite medium Green's function for the isotropic scattering is used in the analytical expressions of exit distributions in F_N method. The numerical calculations performed using H_N method. The convergence of our numerical results is better than F_N method.

1. INTRODUCTION

The F_N method was used to solve some problems in neutron transport theory such as half-space albedo problem, the half-space constant source problem, slab albedo problem, the critical slab problem etc. [1-3]. This method was developed from the use of Placzek lemma as C_N method [5-7]. The Placzek lemma states that for a given finite homogeneous medium bounded by a surface S and surrounded by vacuum, the angular flux inside S does not change if i) medium is extended to infinity, ii) neutron sources outside S are removed and sources inside are retained, and iii) an additional source $\hat{\Omega}.\hat{n}_i\Psi(\vec{r},\vec{\Omega})$ is put on the surface S, where $\hat{n}_i$ is a unit vector inside normal to S. The integro-differential form of the one-speed neutron transport equation in plane geometry is given by

$$\mu\frac{\partial\Psi(x,\mu)}{\partial x}+\Psi(x,\mu)=c\int_{-1}^{1}\Psi(x,\mu)f(\mu,\mu')d\mu'+s(x,\mu) \tag{1}$$

Here $\Psi(x,\mu)$ is the neutron angular distribution at any point, $s(x,\mu)$ is the source function, c is the mean number of secondary neutrons per collision, μ is the direction cosine of the propagating neutrons, $f(\mu,\mu')$ is the scattering function and normalized to unity. The infinite medium problem for the half-space is defined by [2,3]

$$\mu\frac{\partial\Psi_1(x,\mu)}{\partial x}+\Psi_1(x,\mu)=c\int_{-1}^{+1}\Psi_1(x,\mu')f(\mu,\mu')d\mu'+s(x,\mu)H(x)+\mu\Psi(0,\mu)\delta(x) \tag{2}$$

where $\mu\Psi(0,\mu)\delta(x)$ represents the sources on the surface at $x=0$ and $H(x)$ is a unit step function which is defined as

$$H(x)=\begin{cases}1 & \text{for} \quad x\in(0,\infty)\\ 0 & \text{otherwise}\end{cases}$$

$$\Psi_1(x,\mu)=H(x)\Psi(x,\mu) \tag{3}$$

To write an analytical expression for $\Psi_1(x,\mu)$, we need to define the infinite medium Green's function. $G(x_0\to x;\mu_0\to\mu)$ represents the infinite medium Green's function which satisfies the following equation [4]

$$\frac{\partial G(x_0 \to x;\mu_0 \to \mu)}{\partial x} + G(x_0 \to x;\mu_0 \to \mu) = c\int_{-1}^{+1} G(x_0 \to x;\mu_0 \to \mu')f(\mu,\mu')d\mu' + \delta(x-x_0)\delta(\mu-\mu_0) \tag{4}$$

subject to

$$G(x_0 \to \pm\infty;\mu_0 \to \mu) = 0 \quad . \tag{5}$$

It means that we seek the distribution from a plane source of unit strength at $x = x_0$ emitting in the direction $\mu = \mu_0$. $G(x_0 \to x;\mu_0 \to \mu)$ is the angular flux at x in the direction μ in an infinite homogeneous medium, produced by a plane unit source at x_0 emitting in the direction μ_0. Then for the half-space problem, $\Psi_1(x,\mu)$ can be written as [2,3]

$$\Psi_1(x,\mu) = \int_{-1}^{1} d\mu_0 \int_{-\infty}^{\infty} dx_0 G(x_0 \to x;\mu_0 \to \mu)\left[H(x_0)s(x_0,\mu_0) + \mu_0\Psi(0,\mu_0)\delta(x_0)\right]. \tag{6}$$

In the extremely anisotropic scattering case, $f(\mu,\mu')$ is defined by

$$f(\mu,\mu') = \frac{a}{2} + b\delta(\mu-\mu') + d\delta(\mu+\mu'). \tag{7}$$

$a = 1-\alpha,\ b = \alpha,\ d = 0$ represents forward scattering with isotropic scattering, $a = 1-\alpha,\ b = 0,\ d = \alpha$ represents backward scattering with isotropic scattering, $\alpha = 0$ corresponds to the isotropic scattering. The infinite medium Green's function for anisotropic scattering case has been obtained in terms of the infinite medium Green's function for the isotropic scattering as [8]

$$\begin{aligned} G_{anis}\left(x_0 \to x;\mu_0 \to \mu\right) &= \frac{p+q}{2q}G_{is,c'}\left(x_0' \to x';\mu_0 \to \mu\right) + \frac{dc}{2q}G_{is,c'}\left(x_0' \to x';\mu_0 \to -\mu\right) \\ &+ \frac{dc}{2q}G_{is,c'}\left(x_0' \to x';-\mu_0 \to \mu\right) + \frac{p-q}{2q}G_{is,c'}\left(x_0' \to x';-\mu_0 \to -\mu\right) \end{aligned} \tag{8}$$

in Eq. (8), $G_{is,c'}$ is replaced by

$$\begin{aligned} G^{\mp}_{is,c'}\left(x_0' \to x';\mu_0 \to \mu\right) &= \frac{\phi'(\mp\nu_0',\mu)\,\phi'(\mp\nu_0',\mu_0)}{N(\nu_0')}\exp\left(\frac{|x'-x_0'|}{\nu_0'}\right) \\ &+ \int_0^1 \frac{\phi'(\mp\nu',\mu)\,\phi'(\mp\nu',\mu_0)}{N(\nu')}\exp\left(\frac{|x'-x_0'|}{\nu'}\right)d\nu' \quad , \end{aligned} \tag{9}$$

where the upper sign is for $x < x_0$, the lower sign is for $x > x_0$ and

$$x' = xq \quad , \quad \nu_0' = q\nu_0 \quad , \quad p = 1-bc \quad , \quad q = \sqrt{p^2 - d^2c^2} \quad , \quad c' = \frac{ac}{1-bc-dc} \quad . \tag{10}$$

$\pm\nu_0'$ are the two discrete eigenvalues with eigenfunctions $\phi'(\pm\nu_0',\mu)$ and $\pm\nu'$ are the continuum eigenvalues on the range [-1,+1] with eigenfunctions $\phi'(\pm\nu',\mu)$.

$$\phi'(\pm\nu_0',\mu) = \pm\frac{c'\nu_0'}{2}\frac{1}{\pm\nu_0'-\mu} \quad , \quad \phi'(\nu',\mu) = \frac{c'\nu'}{2}P\frac{1}{\nu'-\mu} + \lambda(\nu')\delta(\nu'-\mu) \tag{11}$$

with

$$\lambda(\nu') = 1 - c'\nu'\tanh^{-1}\nu'. \tag{12}$$

The normalization constants corresponding to discrete and continuum modes are respectively

$$\begin{aligned} N'(\pm\nu_0') &= \int_{-1}^{1}\mu\phi'(\pm\nu_0')\phi(\pm\nu_0')d\mu = \pm\frac{c'\nu_0'^3}{2}\left[\frac{c'}{\nu_0'^2-1}-\frac{1}{\nu_0'^2}\right] \\ N'(\nu') &= \int_{-1}^{1}\mu\phi'(\pm\nu')\phi(\pm\nu')d\mu = \nu'\left\{\left(1-c'\nu'\tanh^{-1}\nu'\right)^2 + \frac{c'^2\pi^2\nu'^2}{4}\right\} \end{aligned} \tag{13}$$

2. THE CRITICAL SLAB PROBLEM

Consider a slab extending from $-\tau \le x \le \tau$. It is surrounded by vacuum and there is no external sources. We seek the critical thickness τ for $c' > 1$ [9-18]. The exit distributions at $x = \pm\tau$ are given by [2,3]

$$\begin{aligned} \Psi(-\tau,-\mu) &= -\int_0^1 d\mu_0 G\left(-\tau\to-\tau^+;-\mu_0\to-\mu\right)\mu_0\Psi(-\tau,-\mu_0) \\ &+\int_0^1 d\mu_0 G\left(-\tau\to-\tau^+;\mu_0\to-\mu\right)\mu_0\Psi(-\tau,\mu_0)+\int_0^1 d\mu_0 G\left(\tau\to-\tau^+;-\mu_0\to-\mu\right)\mu_0\Psi(\tau,-\mu_0) \\ &-\int_0^1 d\mu_0 G\left(\tau\to-\tau;\mu_0\to-\mu\right)\mu_0\Psi(\tau,\mu_0) \quad , \qquad \mu>0 \end{aligned} \tag{14a}$$

$$\begin{aligned} \Psi(\tau,\mu) &= -\int_0^1 d\mu_0 G\left(-\tau\to-\tau^+;-\mu_0\to\mu\right)\mu_0\Psi(-\tau,-\mu_0) \\ &+\int_0^1 d\mu_0 G\left(-\tau\to\tau^-;\mu_0\to\mu\right)\mu_0\Psi(-\tau,\mu_0)+\int_0^1 d\mu_0 G\left(\tau\to\tau^-;-\mu_0\to\mu\right)\mu_0\Psi(\tau,-\mu_0) \\ &-\int_0^1 d\mu_0 G\left(\tau\to\tau^-;\mu_0\to\mu\right)\mu_0\Psi(\tau,\mu_0) \quad , \qquad \mu>0 \end{aligned} \tag{14b}$$

with

$$s(x,\mu) = 0 \tag{15}$$

The boundary conditions are

$$\Psi(\tau,-\mu) = \Psi(-\tau,\mu) = 0 \quad , \quad \mu>0 \tag{16}$$

We use the following expansions in Eq.(14a)

$$\Psi(-\tau,-\mu) = \Psi(\tau,\mu) = \sum_{l=0}^{N} a_\ell\mu^\ell \quad , \quad \mu>0 \tag{17}$$

Here $\mathrm{H_N}$ method is used in the numerical calculations, it means that equation (14a) is multiplied by μ^{m+1} ($m = 0,1,2...$) and integrated over $\mu \in [0,1]$ using Eqs.(16-17) and we obtain [9]

$$\sum_{\ell=0}^{N} a_\ell \left\{ \frac{2q}{m+\ell+2} + (p+q)\left[\frac{c'^2 \nu_0'^2}{4N'(\nu_0')} A_\ell(\nu_0') \left(A_m(\nu_0') + B_m(\nu_0') \exp(-\frac{2q\tau}{\nu_0'}) \right) + \frac{c'^2}{4}(I_{\ell m}^{AA} + I_{\ell m}^{ABE}) \right] \right.$$

$$+dc\left[\frac{c'^2 \nu_0'^2}{4N'(\nu_0')} A_\ell(\nu_0') \left(B_m(\nu_0') + A_m(\nu_0') \exp(-\frac{2q\tau}{\nu_0'}) \right) + \frac{c'^2}{4}(I_{\ell m}^{AB} + I_{\ell m}^{AAE}) \right]$$

$$+dc\left[\frac{c'^2 \nu_0'^2}{4N'(\nu_0')} B_\ell(\nu_0') \left(A_m(\nu_0') + B_m(\nu_0') \exp(-\frac{2q\tau}{\nu_0'}) \right) + \frac{c'^2}{4}(I_{m\ell}^{AB} + I_{\ell m}^{BBE}) \right]$$

$$\left. +(p-q)\left[\frac{c'^2 \nu_0'^2}{4N'(\nu_0')} A_\ell(\nu_0') \left(B_m(\nu_0') + A_m(\nu_0') \exp(-\frac{2q\tau}{\nu_0'}) \right) + \frac{c'^2}{4}(I_{\ell m}^{AB} + I_{\ell m}^{AAE}) \right] \right\} = 0 \qquad (18)$$

For $c' > 1$, ν_0' is replaced by $-i\nu_0'$ in Eq.(18) and the following expressions are used

$$A_\ell(-i\nu_0') = \frac{1}{\ell+1} + i\nu_0' A_{\ell-1}(-i\nu_0') \qquad (19a)$$

$$A_0(\nu_0') = 1 + i\nu_0' \log\left(1 + \frac{i}{\nu_0'}\right) = 1 - \frac{1}{c} + \frac{i\nu_0'}{2}\log\left(1 + \frac{1}{\nu_0'^2}\right) \qquad (19b)$$

$$B_\ell(-i\nu_0) = -i\nu_0' B_{\ell-1}(-i\nu_0') - \frac{1}{\ell+1} \qquad (19c)$$

$$B_0(-i\nu_0') = \frac{2}{c'} - 1 + i\nu_0' \log\left(1 + \frac{i}{\nu_0'}\right) = \frac{1}{c} - 1 + \frac{i\nu_0'}{2}\log\left(1 + \frac{1}{\nu_0'^2}\right) \quad . \qquad (19d)$$

We have $N+1$ homogenous linear algebraic equations in Eq. (18) and τ is calculated numerically for different values of c'.

Table 1. The critical slab thickness for forward and backward scattering

N	$c' = 1.2$ $\alpha = 0.4545$	$c' = 1.3$ $\alpha = 0.6061$	$c' = 1.5$ $\alpha = 0.7273$	$c' = 2.0$ $\alpha = 0.8182$	$c' = 3.83$ $\alpha = 0.8770$
Backward Scattering					
0	3.50127	3.24863	2.95828	2.49339	1.12438
1	3.49223	3.23830	2.95415	2.55425	1.86660
2	3.49526	3.24275	2.95959	2.55489	1.88612
3	3.49534	3.24303	2.96073	2.55853	1.88520
4	3.49534	3.24303	2.96075	2.55923	1.88767
5	3.49534	3.24303	2.96075	2.55926	1.88961
6	3.49534	3.24303	2.96075	2.55926	1.89030
7	3.49534	3.24303	2.96075	2.55926	1.89051
8	3.49534	3.24303	2.96075	2.55926	1.88995
9	3.49534	3.24303	2.96075	2.55926	1.89026
10	3.49534	3.24303	2.96075	2.55926	1.89026
Ref.(10)	F_8 3.4953	F_4 3.2428	F_4 2.9604	F_8 2.5587	F_{10} 1.8907
Forward Scattering					
0	5.17180	5.61315	5.41242	6.73837	8.38399
1	5.14937	5.60634	5.98773	6.30868	6.06111
2	5.15719	5.62441	6.03244	6.31828	5.65540
3	5.15752	5.62633	6.04951	6.24703	5.66119
4	5.15752	5.62635	6.05056	6.22312	5.66944
5	5.15752	5.62635	6.05056	6.22049	5.66582
6	5.15752	5.62635	6.05056	6.22051	5.66219
7	5.15752	5.62635	6.05056	6.22052	5.66083
8	5.15752	5.62635	6.05056	6.22052	5.66061
9	5.15752	5.62635	6.05056	6.22052	5.66065
10	5.15752	5.62635	6.05056	6.22052	5.66069
Ref.(10)	F_8 5.1575	F_8 5.6263	F_9 6.0505	F_9 6.2203	F_{10} 5.6590

3. CONCLUSIONS

We have considered the critical slab problems with extremely anisotropic scattering. The analytical expressions of exit distributions in F_N method are used for each problem. The numerical results are obtained from the exit distributions which are multiplied by μ^{m+1} ($m = 0,1,2...$) and integrated over $\mu \in [0,1]$, [9]. This is a different numerical calculation from F_N method. In Table (1), the critical slab thickness are obtained for different values of c' and α. These values were convergence at different order of N when the F_N method used as in Ref. [10].

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