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ON SECOND ORDER CORRECTION TO THE AVERAGE EXTRAPOLATION LENGTH

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A B S T R A C T

A first-order expression for the average extrapolation length in bare thermal reactors was given previously on the basis of a many-group calculation. C. Yalçın extended this calculation to second order and obtained a complicated correction term which does not seem easy to apply. His results are analyzed and simplified here by considering the limit for continuous slowing down. It is shown that the second order correction is of the order of $\frac{\bar{d}}{R}$, where $\bar{d}$ is the first order average extrapolation length and R is a length which characterizes the size of the reactor.

On Second Order Correction to the Average Extrapolation Length

E. İnönü*

Introduction

An expression for the average extrapolation length ($\overline{d}$) to be used in determining the critical size of bare thermal reactors is given in the book of A.M.Weinberg and E.P.Wigner[1] (page 475). It reads as:

$$\overline{d} = 0.71 \frac{\int D(E)\varphi(E)\frac{dE}{\Sigma_{tr}(E)} + D_t \frac{\lambda_t}{\Sigma_{tr,t}}}{\int D(E)\varphi(E)\,dE + D_t \lambda_t} \qquad (1)$$

where $D(E)$ is the diffusion coefficient, $\Sigma_{tr}(E)$ is the macroscopic transport cross-section, $\varphi(E)$ is the energy dependent factor in the expression for the critical flux in the reactor, all of which refer to non-thermal neutrons and D_t, $\Sigma_{tr,t}$, λ_t are the corresponding quantities for thermal neutrons. As mentioned by Weinberg and Wigner this formula is a first order approximation and gives in general too large an extrapolation distance. We would like to indicate here the form of the second order correction obtained by the same method[2] which was used to derive the first order formula. It will be seen that the correction is of the order of $\frac{\overline{d}}{R}$ where R is a length characterizing the size of the reactor (e.g. the radius for a sphere-shaped reactor).

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*On leave from the Middle East Technical University, Ankara.

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(1) A.M.Weinberg-E.P.Wigner, The Physical Theory of Neutron Chain Reactors, p.475, The University of Chicago Press, Chicago, 1958.

(2) E.İnönü, Proceedings of the Second Geneva Conference on Peaceful Uses of Atomic Energy, Vol 16, p.701 (1958).

The method-many group theory-consists of considering n groups of neutrons, thus allowing the transport mean free path to be different in every group, making use of all the solutions (ie, the higher modes) of Laplace's equation to express the continuity of the flux at the boundary between successive groups and expressing slowing down in any group by the appropriate kernel which leads to a non-escape probability for every mode in this group according to the second fundamental theorem. Thus a solution $\psi_{i,\alpha}$ of the equation

$$\nabla^2 \psi_{i,\alpha} + B^2_{i,\alpha} \psi_{i,\alpha} = 0 \qquad (2)$$

for the i'th group is multiplied by the non-escape probability $\kappa_{i,\alpha}(B^2_{i,\alpha}; E_i, E_{i+1})$ in slowing down in this group. $B^2_{i,\alpha}$ is the eigen value corresponding to the α'th mode, determined by the "linear extrapolation " condition; it varies from group to group through its dependence on the extrapolation length $d_i = \frac{0.71}{\Sigma_{tr\,i}} \approx 0.71\lambda_i$ where λ_i is the transport mean free path for the i'th group. For instance for a sphere $B_{i,\alpha}$ is given by the α 'th root of the equation

$$\tan B_{i,\alpha} R = -\frac{0.71 \lambda_i B_{i,\alpha}}{1 - \frac{0.71\lambda_i}{R}} \qquad (3)$$

For quick estimates one may also use the values of $B_{i,\alpha}$ given by the simpler boundary-condition of vanishing flux on the extrapolated surface.

The transition from one group to the next is carried out by expressing any $\Psi_{i,\alpha}$ in terms of the eigen functions $\Psi_{i+1,\beta}$ of the successive group as

$$\Psi_{i+1,\alpha} = (R_i)_{\alpha\beta}\, \Psi_{i,\beta} \qquad (4)$$

The matrices R_i defined in this way are orthogonal and satisfy the equation

$$R_n R_{n-1} \cdots R_2 R_1 = I \quad \text{(the identity matrix)} \qquad (5)$$

It is important to know the order of magnitude of the matrix elements $R_{i,1\mu}$ and $R_{i,11}$. Taking various simple geometries one can see that $R_{i,1\mu}$ contains the factor $\frac{\lambda_i - \lambda_{i+1}}{R}$ and hence is at most of the order of $\frac{\lambda}{R}$ where λ is the largest value of the transport mean free path, while $R_{i,11}$ differs from 1 by a term of second order in $\frac{\lambda}{R}$. For the sphere e.g. one finds:

$$R_{i,1\mu} = -2 \times 0.71 \; \frac{\mu}{(\mu^2-1)(1+\beta_i^2\mu^2)^{1/2}} \; \frac{\lambda_i - \lambda_{i+1}}{R} \qquad (6)$$

where $\beta_i = \frac{0.71\lambda_i}{R}$. Further details about this model may be found in the article[2] quoted above.

Review of First-order solution. In this formalism, neglecting resonance absorption and fast fission the criticality may be expressed by the equation:

$$\text{Det}\ (kM-1) = 0 \qquad (7)$$

where k is the infinite medium multiplication factor (ηf in this case) and M is the matrix

$$M = R_n K_n R_{n-1} K_{n-1} \cdots R_1 K_1 \qquad (8)$$

where R_i is the matrix defined by (4) and K_i is a diagonal matrix the α'th element of which is equal to the non-escape probability $K_{i,\alpha}$ for α 'th mode in the i'th group.

To first order in $\frac{\lambda}{R}$, the equation (7) reduces to:

$$k\, K_{1,1}(\lambda_1)\, K_{2,1}(\lambda_2) \cdots K_{n,1}(\lambda_n) = 1 \qquad (9)$$

Equating the left-hand side to $k\, K_{1,1}(\bar{\lambda}) K_{2,1}(\bar{\lambda}) \cdots K_{n,1}(\bar{\lambda})$ and solving for $\bar{\lambda}$, the average extrapolation length, one obtains to first order:

$$\bar{\lambda}_1 = \frac{\sum_{i=1}^{n} \left(\frac{\partial \ln K_{i,1}}{\partial \lambda_i}\right)_{\lambda=0} \lambda_i}{\sum_{i=1}^{n} \left(\frac{\partial \ln K_{i,1}}{\partial \lambda_i}\right)_{\lambda=0}} \qquad (10)$$

We note that $\left(\frac{\partial \ln K_{i,1}}{\partial \lambda_i}\right)_{\lambda=0} = \left(\frac{\partial \ln K_{i,1}}{\partial B_{i,1}}\right)_{\lambda=0} \left(\frac{\partial B_{i,1}}{\partial \lambda_i}\right)_{\lambda=0}$ and $\left(\frac{\partial B_{i,1}}{\partial \lambda_i}\right)_{\lambda_i=0}$ has the same value for all groups. Dropping the indices when we refer to the usual buckling we can say therefore that in this model the weight factor for the appropriate average over groups turns out to be

proportional to the value of $\left(\frac{\partial \ln K}{\partial B}\right)_{\lambda=0}$ for the group.

A weight factor which involves the flux may also be obtained by bringing in the diffusion theory explicitly and considering the non-escape probability for an infinitesimal energy interval. The average flux loss through the surface while slowing down by dE may be expressed as

$$DB^2\left(\int \phi dV\right)\frac{dE}{\xi\Sigma_s E} \quad , \tag{11}$$

since the flux lost per second is $-v D\int \vec{\nabla}\phi \cdot d\vec{S} = v DB^2 \int \phi dV$ and the time spent in slowing down by dE is on the average $dt = \frac{1}{v}\frac{dE}{\xi\Sigma_s E}$. The non-escape probability for the infinitesimal interval becomes

$$K = 1 + \frac{DB^2 dE}{\xi\Sigma_s E} \tag{12}$$

where $dE < 0$. We thus obtain

$$\frac{\partial \ln K}{\partial B} = \frac{2DB}{\xi\Sigma_s E}dE \tag{13}$$

Introducing the age for the interval in question $d\tau = \frac{D(E)}{\xi(E)\Sigma_s(E)E}$ we see that the weight factor is proportional to the contribution of the interval to the migration area. Assuming that the energy spectrum can be represented with the asymptotic expression $\varphi(E) \propto \frac{1}{\xi\Sigma_s E}$ (no absorption, no leakage) one may also say that the weight factor is proportional to $D(E)\varphi(E)dE$ This last expression involves only the

existing energy spectrum in the reactor and consequently seems to have more general validity. In fact Weinberg and Wigner claim that it is more nearly correct when the effects of absorption and leakage on the energy spectrum are taken into account, the fast effect being however always neglected.

Second order solution. In the present model the second order correction to $\bar{d}$ will be obtained by solving the equation (7) to second order. This has been done by C.Yalçın[3] in the following way. Developing the determinant he obtains

$$k M_{11} - 1 = k^2 \sum_{\mu=2}^{\infty} \frac{M_{1\mu} M_{\mu 1}}{k M_{\mu\mu} - 1} \qquad (14)$$

as the second order equation. Expressing M_{11}, $M_{1\mu}$ and $M_{\mu\mu}$ in terms of the matrix elements $K_{i,\alpha}$ and $R_{i,\alpha\beta}$, the equation is transformed into the form:

$$k K_{1,1} K_{2,1} \cdots K_{n,1} (1 + \Lambda) = 1 \qquad (15)$$

where Λ is a complicated second order term which depends not only on the moderation and diffusion properties of the medium through the K's, but also on the explicit shape of the reactor through the R's. It is given by:

$$1 + \Lambda = \prod_{i=1}^{n} R_{i,11} + \sum_{i=1}^{n} \sum_{\mu=2}^{\infty} R_{i,1\mu} R_{i,\mu 1} (\pi_\mu + \pi_\mu^2 + \cdots) \qquad (16)$$
$$+ \sum_{i=1}^{n} \sum_{\mu=2}^{\infty} \{ R_{i+1,1\mu} N_{i+1,\mu} R_{i,\mu 1}$$
$$+ R_{i+2,1\mu} N_{i+2,\mu} N_{i+1,\mu} R_{i,\mu 1} + \cdots$$
$$+ R_{i+n-1,1\mu} N_{i+n-1,\mu} \cdots N_{i+1,\mu} R_{i,\mu 1} \} (1 + \pi_\mu + \pi_\mu^2 + \cdots)$$

(3) C.Yalçın: Master Thesis, University of Ankara, October 1961.

where $\Pi_\mu = k\, K_{n,\mu} \cdots K_{1,\mu}$

$$N_{i,\mu} = \frac{K_{i,\mu}}{K_{i,1}} \qquad (17)$$

Again, equating the left-hand side of (15) to $\frac{1}{2} K_{i,1}(\lambda)$ C.Yalçın obtains for the average extrapolation length, the following expression which is valid up to second order in $\frac{\lambda}{D}$

$$\bar{\lambda} = \bar{\lambda}_1 + \frac{1}{2} \frac{\sum_{i=1}^{n} (\lambda_i^2 - \lambda_1^2) \left(\frac{\partial^2 \ln K_{i,1}}{\partial \lambda_i^2}\right)_{\lambda=0}}{\sum_{i=1}^{n} \left(\frac{\partial \ln K_{i,1}}{\partial \lambda_i}\right)_{\lambda=0}} + \frac{\Lambda}{\sum_{i=1}^{n} \left(\frac{\partial \ln K_{i,1}}{\partial \lambda_i}\right)_{\lambda=0}} \qquad (18)$$

Discussion. a) The third term of (18). In order to estimate the importance of the term Λ in this expression we consider the limiting case for $n \to \infty$ and $\Delta\lambda = \lambda_i - \lambda_{i+1} \to 0$, ie when the intervals are made infinitesimally small. As Λ depends on the matrix elements $R_{i,\mu}$ which come in because of the discontinuous change in λ_i from group to group, it is plausible to expect that it would vanish when the group intervals are decreased without limit. In fact $R_{i,\mu}$ contains the factor $\Delta\lambda$ so that a term which contains $R_{i,\mu}\, R_{j,\mu}$ is of second order in $\Delta\lambda$. All such terms in (16) vanish in the limit for $n \to \infty$, $\Delta\lambda \to 0$, since $\lim_{\substack{n\to\infty \\ \Delta\lambda\to 0}} (n\,\Delta\lambda) = \lambda_1 - \lambda_n$ There remains the term $\prod_{i=1}^{n} R_{i,11}$; but from orthogonality of R, one can easily see that $R_{i,11}$ may be expressed to second order as:

$$R_{i,11} = 1 - \frac{1}{2} \sum_{\mu=2}^{\infty} R_{i,\mu}\, R_{i,\mu} \qquad (19)$$

Hence $$\lim_{\substack{n\to\infty \\ \Delta\lambda\to 0}} R_{i,11} = 1 \quad . \tag{20}$$

This completes the proof of our expectation that

$$\lim_{\substack{n\to\infty \\ \Delta\lambda\to 0}} \Lambda = 0 \quad . \tag{21}$$

Therefore it should always be possible to make the effect of Λ on $\bar{\lambda}$ negligible by increasing the number of groups. In particular Λ must not appear at all when the summation for the average is replaced by an integral.

As Λ vanishes, the original equation (7) reduces to (9). It may be worth pointing out that the equation (9) is true to all orders in $\frac{\lambda}{R}$; the non-diagonal elements of the determinant always bring in a factor of the order of $(\Delta\lambda)^2$ and hence vanish in the limit for $n \to \infty$.

b) The second term in (18). This is the term

$$\bar{\lambda}_2 = \frac{\sum_{i=1}^{n} \left(\frac{\partial^2 \ln k}{\partial \lambda^2}\right)_{\lambda=0} (\lambda_i^2 - \bar{\lambda}_1^2)}{\sum_{i=1}^{n} \left(\frac{\partial \ln k}{\partial \lambda}\right)_{\lambda=0}}$$

It comes from the equation (9). Using the diffusion-theoretic expression

$$k = 1 - B^2 d\tau$$

We obtain

$$\bar{\lambda}_2 = \frac{\left(\frac{\partial^2 B^2}{\partial \lambda^2}\right)_{\lambda=0}}{2\left(\frac{\partial B^2}{\partial \lambda}\right)_{\lambda=0}} \frac{\sum_{i=1}^{n} \Delta\tau_i (\lambda_i - \bar{\lambda}_1^2)}{\sum_{i=1}^{n} \Delta\tau_i} \quad . \tag{22}$$

The first factor here is a negative constant which is determined by the shape of the reactor, let us denote it by $-\alpha$; e.g. for the sphere it is given by $\alpha = \frac{3\times 0.71}{2R} \sim \frac{1}{R}$ using the simpler boundary condition of zero flux on the extrapolated surface.

We note that $\bar{\lambda}_2$ may also be written as :

$$\bar{\lambda}_2 = -\alpha \frac{\sum_{i=1}^{n} \Delta\tau_i (\lambda_i - \bar{\lambda}_1^2)}{\sum_{i=1}^{n} \Delta\tau_i} \qquad (23)$$

which shows that it is always negative. Hence the value of the average extrapolation length is decreased by the second order correction (which agrees with the statement in Ref.(1))

The magnitude of $\bar{\lambda}_2$ will be of the order of $\alpha\bar{\lambda}_1 \sim \frac{\bar{\lambda}_1}{R}$. For a critical sphere of enriched $U^{235} - H_2O$ at minimum critical volume, the values of R $\sim$ 20 cm and $\bar{d}_1 \sim$ 3 cm are given in Ref.(1). Therefore in this case $\frac{\bar{d}_2}{\bar{d}_1} \sim$ 5%. Such a correction seems beyond the accuracy of present measurements. If it were possible to measure the extrapolation length for critical spheres so accurately, one would see a slight increase in the extrapolation length with radius.

Finally, we would like to point out that as for $\bar{\lambda}_1$, $\bar{\lambda}_2$ may also be expressed as:

$$\bar{\lambda}_2 = -\alpha \frac{\int D(E)\varphi(E)(\lambda^2 - \bar{\lambda}_1^2)\,dE}{\int \Sigma(E)\varphi(E)\,dE} \qquad (24)$$

and it is plausible to expect that in this form $\bar{\lambda}_2$ will be more nearly correct when absorption and leakage are taken into account (excluding again fast fission).It is therefore the required second order term to be added to the formula given in Ref.(1).

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