THE CRITICAL SLAB PROBLEM FOR LINEARLY ANISOTROPIC SCATTERING WITH THE H_N METHOD

C. Tezcan
Başkent University, Faculty of Engineering, Bağlıca Campus, Eskişehir Yolu 20. Km. 06530, Ankara, Turkey

ABSTRACT

H_N method, is used to calculate the critical slab problem for linearly anisotropic scattering. H_N method is a combination of C_N, F_N and the method of elementary solutions. In C_N method; the integral equations of the Fredholm type, that is; the third form of the transport equation has been used. The integral equations mentioned above have been related with the angular flux at the boundary of the given medium. They have been expressed in terms of the infinite medium Green function. Infinite medium Green function has been obtained by the Fourier transform technique. In F_N method; the integro-differential form of the linear transport equation has been used, the values of eigenvalues have been selected. This selection has been based on an equal spacing scheme for choosing the first two eigenvalues as $(\nu_0, 1)$ and the remaining eigenvalues have been spaced equally in interval $(0,1)$.

In H_N method, the integro-differential form of the transport equation has been used as in F_N method, the eigenvalues have been calculated using the normalization condition of the eigenfunctions and a different numerical procedure has been considered. In this procedure the expansion of the angular flux $\Psi(x,\mu)$ in terms of the eigenfunctions of the method of elementary solutions and their orthogonality relations have been used.

In our work we have used H_N method. We consider the angular flux $\Psi(a,\mu)$, $-1 \le \mu \le 1$ at one of the boundaries of the slab together with the entering flux $\Psi(a,-\mu)$, $0 \le \mu \le 1$. Then, we multiply the angular flux $\Psi(a,\mu)$ with the singular eigenfunctions and integrate over $\mu \in (-1,+1)$, to calculate the expansion coefficients in $\Psi(x,\mu)$. Here the orthogonality relation of the singular eigenfunctions of the method of elementary solutions are used. After obtaining these coefficients we consider the entering angular flux $\Psi(a,-\mu)$, $0 \le \mu \le 1$. We multiply this equation with μ^{m+1} and integrate over $0 \le \mu \le 1$. Thus we get a system of linear algebraic equations. Therefore replacing the appropriate coefficients into the linear algebraic equations the final critically condition is obtained.

It is shown that the method leads to concise equations and accurate numerical results even in the lowest order of approximations.

1. INTRODUCTION

The criticality problems in different geometries for isotropic and anisotropic scattering kernels have been widely investigated in the literature [1-3]. The C_N method [4], using the third form of the transport equation has been also considered to calculate the critical size in different geometries [5-7]. The problem is also considered by Kaşkaş [8] using the third from of the linear transport equation. In our recent works, the H_N method has been used to calculate the half space, finite slab and the criticality problems [9-10]. The purpose of the present work is to solve the critical slab problem.

2. THE CRITICAL SLAB PROBLEM FOR LINEARLY ANISOTROPIC SCATTERING

We consider an homogeneous slab of half thickness a. The one-speed time independent linear transport equation for linearly anisotropic scattering is, [1],

$$\mu \frac{d\Psi}{dx} + \psi(x,\mu) = \frac{c}{2} \int_{-1}^{1} \psi(x,\mu')(1 + 3f\mu\mu')d\mu' \qquad (1)$$

The solution of Eq.(1) is given as

$$\psi(x,\mu) = A(\nu_0)\phi(\nu_0,\mu)\exp\left(-\frac{x}{\nu_0}\right) + A(-\nu_0)\phi(-\nu_0,\mu)\exp\left(\frac{x}{\nu_0}\right)$$

$$+\int_0^1 A(\nu)\phi(\nu,\mu)\exp\left(-\frac{x}{\nu}\right)d\nu+\int_0^1 A(-\nu)\phi(-\nu,\mu)\exp\left(\frac{x}{\nu}\right)d\nu \tag{2}$$

Here $A(\pm\nu_0)$ and $A(\pm\nu)$ represent the expansion coefficients, $\psi(x,\mu)$ angular distribution at any point in the medium $\phi(\pm\nu_0,\mu)$ and $\phi(\pm\nu,\mu)$ are the discrete and continuous eigenfunctions corresponding to discrete and continuum eigenvalues respectively.

$$\phi(\pm\nu_0,\mu)=\frac{c\nu_0}{2}\frac{d(\pm\nu_0\mu)}{\nu_0\pm\mu} \tag{3}$$

$$\phi(\nu,\mu)=\frac{c\nu}{2}P\frac{d(\nu\mu)}{\nu-\mu}+\lambda(\nu)\delta(\nu-\mu) \tag{4}$$

with

$$d(xy)=1+3f(1-c)xy \tag{5}$$

P presents the Cauchy principal value. The eigenfunctions Eqs.(3) and (4) are subject to the following full-range orthogonality relations.

$$\int_{-1}^{1}\mu\phi^2(\pm\nu_0,\mu)d\mu=N(\pm\nu_0) \tag{6}$$

$$\int_{-1}^{1}\mu\phi(\nu,\mu)\phi(\nu',\mu)d\mu=N(\nu)\delta(\nu-\nu') \tag{7}$$

where

$$N(\nu_0)=\frac{c\nu_0^2}{2}(1+r\nu_0^2)\left[\frac{c(1+r\nu_0^2)}{\nu_0(\nu_0^2-1)}-\frac{(1-c)(1+3r\nu_0^2)}{\nu_0(1+r\nu_0^2)}\right] \tag{8}$$

$$N(\nu)=\nu\left[(1+r\nu^2)(1-c\nu\tanh^{-1}\nu)-r(1-c)\nu^2\right]^2+\frac{1}{4}c^2\nu^3\pi^2(1+r\nu^2)^2 \tag{9}$$

with

$$r=3f(1-c) \tag{10}$$

The solution of Eq.(1) is given in Eq.(2). It is clearly seen that the form of Eq.(2) is the same with the corresponding equation for isotropic scattering. The difference comes from the form of the discrete and continuous eigensolutions as shown in Eq.(3-5) with the normalization conditions given by Eqs(6-7). As $f\to 0$ all of these equations reduce to the one with isotropic scattering.

From the symmetry condition

$$\psi(x,\mu)=\psi(-x,-\mu) \tag{11}$$

It follows that

$$A(\nu_0)=A(-\nu_0) \tag{12}$$

$$A(\nu)=A(-\nu) \tag{13}$$

then equation (2) takes the from

$$\psi(x,\mu)=A(\nu_0)\left[\phi(\nu_0,\mu)\exp\left(-\frac{x}{\nu_0}\right)+\phi(-\nu_0,\mu)\exp\left(\frac{x}{\nu_0}\right)\right]$$
$$+\int_0^1 A(\nu)\left[\phi(-\nu,\mu)\exp\left(\frac{x}{\nu}\right)+\phi(\nu,\mu)\exp\left(-\frac{x}{\nu}\right)\right]d\nu \tag{14}$$

$$\psi(a,-\mu)=0 \qquad \text{for } \mu>0 \tag{15}$$

$$\psi(-a,\mu)=0 \qquad \text{for } \mu>0 \tag{16}$$

The flux at $x = a$ is

$$\psi(a,\mu) = A(\nu_0).\left[\phi(\nu_0,\mu)\exp\left(-\frac{a}{\nu_0}\right) + \phi(-\nu_0,\mu)\exp\left(\frac{a}{\nu_0}\right)\right]$$

$$+\int_0^1 A(\nu)\left[\phi(-\nu,\mu)\exp\left(\frac{a}{\nu}\right) + \phi(\nu,\mu)\exp\left(-\frac{a}{\nu}\right)\right]d\nu \quad \text{for } 1 \le \mu \le -1 \tag{17}$$

$\psi(-a,\mu)$ for $\mu > 0$ represents the entering flux at $x = -a$.

First we want to calculate the unknown coefficients $A(\nu_0)$ and $A(\nu)$ in the expansion of the angular flux. To do this we multiply Eq.(17) by $\mu\phi(\nu_0,\mu)$ respectively and integrate over $\mu \in [-1,1]$. The full range orthogonality conditions in Eqs.(6-7) lead to

$$A(\pm\xi) = \frac{\exp\left(-\frac{a}{\xi}\right)}{N(\xi)}\left[-\int_0^1 \mu\phi(\xi,-\mu)\psi(a,\mu)d\mu + \int_0^1 \mu\phi(\xi,\mu)\psi(a,-\mu)d\mu\right] \tag{18}$$

where

$\xi = \pm\nu_0$ or $\pm\nu$.

Assuming the expansions

$$\psi(a,\mu) = \psi(-a,-\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha, \ \mu > 0, \tag{19}$$

for the leaving angular fluxes at the faces $\pm a$, and using Eqs.(15-16), the expansion coefficients in Eq.(18) become

$$A(\xi) = \frac{e^{-\frac{a}{\xi}}}{N(\xi)}\sum_{\alpha=0}^{N} a_\alpha\left[-\int_0^1 \mu^{\alpha+1}\phi(\nu_0,-\mu)d\mu\right]. \tag{20}$$

Defining

$$A_\alpha(\xi) = \frac{2}{c\xi}\int_0^1 \mu^{\alpha+1}\phi(-\xi,\mu)d\mu, \tag{21}$$

$$B_\alpha(\xi) = \frac{2}{c\xi}\int_0^1 \mu^{\alpha+1}\phi(\xi,\mu)d\mu, \tag{22}$$

we have

$$A_0(\xi) = \left[1 - \xi\log\left(1+\frac{1}{\xi}\right)\right](1+3f(1-c)\xi^2) - \frac{3}{2}f(1-c)\xi, \tag{23}$$

$$B_0(\xi) = \frac{2}{c} - \left[1 + \xi\log\left(1+\frac{1}{\xi}\right)\right](1+3f(1-c)\xi^2) + \frac{3}{2}(4\xi-1)f(1-c)\xi, \tag{24}$$

$$A_\ell(\xi) = \frac{1}{\ell+1} - \xi A_{\ell-1}(\xi) - \frac{3f(1-c)\xi}{\ell+2}, \tag{25}$$

$$B_\ell(\xi) = \xi B_{\ell-1}(\xi) - \frac{1}{\ell+1} - \frac{3f(1-c)\xi}{\ell+2}. \tag{26}$$

The angular flux at the boundary of the given problem as is used both in F_N and C_N methods is expanded in powers of μ. The coefficients satisfying the recurrence relations in Eqs.(25) and (26) may easily be calculated. As is seen from numerical results we obtain exact values in the lowest order of approximations. That is the expansion is appropriate and leads to quite good results.

The definitions in Eqs.(21) and (22) make it possible to express Eq.(20) as

$$A(\xi) = \sum_{\alpha=0}^{N} a_\alpha \frac{c\xi}{2} \frac{e^{-\frac{a}{\xi}}}{N(\xi)} [- A_\alpha(\xi)] \tag{27}$$

Now, we return back to equation (17), we multiply the entering flux $\psi(-a,\mu)$ for $\mu > 0$ with μ^{m+1} and integrate over $\mu \in (0,1)$, where m is an integer (m = 0, 1 …). After some algebra considering Eqs.(15), (16), (21), (22) we obtain

$$\frac{cv_0}{2}\exp\left(\frac{a}{v_0}\right)A(v_0)B_m(v_0) + \frac{c}{2}\int_0^1 \frac{v}{N(v)} A(v)A_m(v)\exp\left(-\frac{a}{v}\right)dv$$

$$+\frac{c}{2}\int_0^1 \frac{v}{N(v)} A(v)B_m(v)\exp\left(\frac{a}{v}\right)dv = 0 \tag{28}$$

Replacing the expansion coefficients given in Eq.(27) into Eq.(28) we get

$$\sum_{\alpha=0}^{N} a_\alpha \left\{ \frac{v_0^2}{N(v_0)} \left[A_m(v_0)A_\alpha(v_0)\exp\left(-\frac{2a}{v_0}\right) + B_m(v_0)A_\alpha(v_0) \right] \right.$$

$$\left. + \left[\int_0^1 \frac{v^2}{N(v)}\exp\left(-\frac{2a}{v}\right)A_m(v)A_\alpha(v)dv + \int_0^1 \frac{v^2}{N(v)} B_m(v)A_\alpha(v)dv \right] \right. \tag{29}$$

This equation can be written in short as

$$\sum_{\alpha=0}^{N} a_\alpha L_{m\alpha} = 0, \tag{30}$$

where $L_{m\alpha}$ is the coefficient of the sum in Eq.(29). In zeroth order approximation α = 0 the criticality equations becomes.

$$L_{00} = 0 \tag{31}$$

In the criticality problems it is well known that the discrete eigenvalues v_0 assume purely imaginary values. Therefore we replace v_0 by v_0/i in Eq.(31), then equating the real or imaginary parts of the resulting equation we obtain the criticality condition. That is the relation between secondary neutrons c and critical slab thickness 2a.

In first order approximation, α and m assume values (0,1), leading to

$$a_0 L_{00} + a_1 L_{01} = 0, \quad \text{for} \quad m = 0 \tag{32}$$

$$a_0 L_{10} + a_1 L_{11} = 0, \quad \text{for} \quad m = 1 \tag{33}$$

or

$$\begin{bmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = 0$$

since $a_0 \neq 0$ and $a_1 \neq 0$ then the determinant

$$\begin{bmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{bmatrix} = 0 \tag{34}$$

should be zero.

In N *th* order approximation the determinant

$$\begin{bmatrix} L_{00} & L_{01} & \cdots\cdots & L_{0N} \\ L_{10} & & & \\ L_{20} & & & \\ L_{2N} & & & L_{NN} \end{bmatrix} = 0 \tag{35}$$

should be considered.

3. NUMERICAL RESULTS

In the present work the slab criticality problem for linearly anisotropic scattering is considered. The parameter f represents the contribution of the linearly anisotropic scattering. We consider a slab of half thickness surrounded by vacuum.

In Tables 1 and 2, the numerical values of the critical slab thicknesses are tabulated. These are for the first five approximations for c = 1.1, 2 for isotropic scattering ($f = 0$). We see that these values of the critical thicknesses are in good agreement when they are compared with the half thicknesses for c = 1.1, 1.2 given in tables 3 and 4 and Refs. [5,6,11].

For values of the parameters c, f, corresponding to the average number of secondaries and linearly anisotropic scattering respectively, numerical results of the critical slab thicknesses of this work are listed in Tables 1, 2 and 5. Table 5 corresponds to the critical slab thicknesses under vacuum boundary conditions for different values of f. As N the order of approximation increases the values of the critical slab thicknesses approaches to their exact values. We also note that the results are very close to exact results even in the lowest order of approximations.

4. CONCLUSIONS

In order to check the feasibility of the H_N method for anisotropic problems , the method is applied to the critical slab problem, and the corresponding criticality condition is derived. In the procedure normal mode expansion of the angular distribution with the unknown expansion coefficients is used, these coefficients using the angular distribution at the boundaries of the given problem together with the full range orthogonality conditions are calculated in terms of unknown leaving angular fluxes at the faces $\pm a$. As mentioned above multiplying the entering angular flux at $x = a$ with μ^{m+1} and integrating over μ and replacing the calculated expansion coefficients back in to this equation, we obtain a set of exact equations.

A comparison with the existing literature is indicated in tables (1-5). We observe a nice agreement with the well known numerical results. The method can also be extended to calculate the critical radius and the flux distribution for a sphere and a cylinder of infinite length, together with the two media and multilayered problems.

Table 1. The slab critical thickness for c=1.1 and $f = 0$(Türeci, R.G. 2004, this work)

N	
0	4.22483362
1	4.22661176
2	4.22661933
3	4.22661933
4	4.22661933

Table 2. The slab critical thickness for c=2 and $f = 0$ (Türeci, R.G.2004, this work)

N	
0	0.62187837
1	0.62205449
2	0.62204923
3	0.62205185
4	0.62205180

Table 3. The critical slab half-thickness with isotropic scattering (Siewert, C.E. 2001)

Model	c = 1.1	c = 1.5	c = 2.0
L = 0	2.113310	6.050565(-1)	3.110259(-1)

Table 4. The critical slab half-thickness of C_N method for c=2 for isotropic scattering (Kavenoky, A. 1978)

Approximation	First Kind of Equation	Second Kind of Equation
C_0	0.62187834	-
C_1	0.62205449	0.63080859
C_2	0.62204924	0.63182920
C_3	0.62205183	0.62469777
C_4	0.62205181	0.62231189

Table 5. The critical slab thickness in mfp for c=1.1 (this work)

f	N					
	0	1	2	3	4	5
0.1	4.40040342	4.40246646	4.40247413	4.40247414	4.40247413	4.40247413
0.2	4.60231106	4.60474585	4.60475374	4.60475375	4.60475375	4.60475375
0.3	4.83818984	4.84112771	4.84113601	4.84113602	4.84113602	4.84113602
0.3	0.96116253	0.96141401	0.96137363	0.96137825	0.96137821	0.96137824

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