Turkish Journal of Nuclear Sciences, 8 (1981) 61-67

COMPARISON FUNCTION TECHNIQUES IN MULTIGROUP SPACE DEPENDENT REACTOR STABILITY

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ABSTRACT

In this paper a mathematical model for a bare homogeneous cylindrical reactor core is formulated which accounts for an arbitrary number of prompt and six delayed neutron groups. The feedback is expressed as linear temperature coefficients of the core parameters. The resulting system of weakly coupled non-linear mixed (partial and ordinary) set of differential equations are examined from a stability point of view. With the aid of a generalization of Naguma-Westphal theorem a method of constructing comparison functions are presented. The comparison function can either be used to check the stability of the system under perturbations or to determine bounds for the ensuing transients. Perturbations may be changes (not necessarily small) in the initial distributions or in the system parameters. Knowledge of the equilibrium solutions enables one to test the system for asymptotic stability and also determine a region of stability in the solution space. Furthermore a bound for the rate of convergence to equilibrium becomes obtainable, since the comparison functions bound the true solution from above and below. One can specify the bounds on the changes in the initial distributions in order to get the bounds on the resulting transients or conversely one can limit the transient and obtain the allowable perturbations. Similarly changes in the system parameters can be limited by bounds on the ensuing transients and/or the new equilibrium solutions.

ÖZET

Bu çalışmada çıplak, homojen bir silindirik reaktör kalbi için herhangi sayıda ani ve 6 gecikmiş nötron grubu içeren matematiksel bir model kurulmuş ve geri besleme kalp parametrelerinin lineer sıcaklık katsayıları cinsinden ifade edilmiştir. Elde edilen zayıf biçimde bağımlı, non-lineer, kısmi ve adi karışık diferansiyel denklem sistemi stabilite açısından incelenmiştir. Nagumo-Westphal Lemma'sının genelleştirilmiş bir şekli yardımıyla

kıyaslama fonksiyonları kurulmak üzere bir yöntem önerilmiştir. Bu kıyaslama fonksiyonları sistemin pertürbasyonlar altında stabilitesini incelemekte kullanılabileceği gibi ortaya çıkan transiyentlere sınırlamalar da verebilir. Pertürbasyonlar başlangıç dağılımlarında veya sistem parametrelerinde küçük olması gerekmeyen değişiklikler olabilir. Denge çözümlerinin bilinmesi sistemin asimptotik stabilitesini ve çözüm uzayında stabilite bölgesini belirlemeye yarayabilir. Ayrıca denge çözümüne dönüş hızı için limitler elde edilebilir çünkü kıyaslama fonksiyonları gerçek transiyent çözümü alttan ve üstten sınırlamaktadır. Başlangıç dağılımlarındaki değişimlere sınırlamalar koyarak sonuç transiyentlere alt ve üst limitler bulunabileceği gibi transiyent çözümü sınırlayarak müsaade edilebilecek pertürbasyonlar belirlenebilir. Benzer biçimde sistem parametrelerindeki değişiklikler, sonuç transiyentler veya yeni denge çözümleri için tayin edilmiş limitler yardımıyla sınırlanabilir.

INTRODUCTION

Equations describing the neutronics and temperature distribution are;

$$\left.\begin{aligned}
&[v^{-1}]\frac{\partial\vec{\phi}}{\partial t}=[D]\nabla^2\phi+[B]\vec{\phi}+(1-\beta)\vec{f}'_0\,[F^T]\vec{\phi}+\sum_{i=1}^{6}\lambda_i\vec{f}_i C_i+\vec{S}_0\\
&\frac{\partial C_i}{\partial t}=\beta_i[F^T]\vec{\phi}-\lambda_i C_i \qquad i=1,..,6\\
&\frac{\partial}{\partial t}\left[\rho(T)C(T)T(\vec{r},t)\right]=\nabla.k(T)\nabla T(\vec{r},t)-h\left[T(\vec{r},t)-T_c(\vec{r},t)\right]\\
&\qquad+\gamma[F^T][v^{-1}]\vec{\phi}
\end{aligned}\right\}(1)$$

where; $\rho(T)$, $C(T)$, $k(T)$, γ are the density, heat capacity, coefficient of thermal conductivity as functions of temperature and energy yield per fission event respectively, $\vec{\phi}$ is the vector of neutron group fluxes, f'_0 and f_i is the energy spectrum vector for prompt fission neutrons and delayed neutrons, $\vec{S}_0$ is the external source energy spectrum vector, $[D]$ is the diagonal matrix of group diffusion constants, $[v^{-1}]$ is the diagonal matrix of group reciprocal velocities, $[B]$ is the matrix of group absorption cross sections with negative signs (along the diagonal) and group transfer cross sections (off diagonal terms), $[F^T]$ is the transpose of $\nu\Sigma_f$ (average number of fission neutrons times the group fission cross sections)

vector. h is the product of convective heat transfer coefficient and coolant channel heat transfer area per unit volume.

i.e. $\vec{f}_i \equiv \begin{pmatrix} f_{i_1} \\ \cdot \\ \cdot \\ \cdot \\ f_{i_6} \end{pmatrix}$, $\vec{S}_0 \equiv \begin{pmatrix} S_{0_1} \\ \cdot \\ \cdot \\ \cdot \\ S_{0_G} \end{pmatrix}$, $[D] \equiv \begin{bmatrix} D_1 0 \ldots . 0 \\ 0 D_2 \ldots . 0 \\ \cdot\cdot \\ \cdot\cdot \\ \cdot\cdot \\ \cdot\cdot \\ 0 0 \ldots . D_G \end{bmatrix}$, $[v^{-1}] \equiv \begin{bmatrix} 1/v_1 0 \ldots . 0 \\ 0 \; 1/v_2 \; \ldots 0 \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ 0 \quad 0 \ldots . 1/v_G \end{bmatrix}$

$$[B] = \begin{bmatrix} -\Sigma_{a1} \Sigma_{S21} \ldots . \Sigma_{SG1} \\ \Sigma_{S12} - \Sigma_{a^2} \ldots . \Sigma_{SG2} \\ \Sigma_{S1G} \Sigma_{S2G} \ldots . \Sigma_{aG} \end{bmatrix}, \quad [-T] = \left[(\nu\Sigma_f)_1 , \ldots . , (\nu\Sigma_f)_G \right]$$

Bc ndary conditions are of the first kind: $\vec{\phi}(\vec{R},t) = C_i(\vec{R},t) = 0$, $T(\vec{R},t) = T_a$ where R denotes boundary and T_0 is the specified surface (ambient) temperature. Initial conditions are specified as; $\vec{\phi}(\vec{r}, 0) = \vec{\phi}_0(r)$, $C_i(\vec{r}, 0) = C_{i0}(\vec{r})$, $T(\vec{r}, 0) = T_0(\vec{r})$. where (o) denotes equilibrium distributiers.

FEEDBACK MODEL

Expanding the temperature dependence of the parameters in a Taylor's series around the average fuel temperature retaining the first two terms and some manipulation of the Eqs. 1 result in the component form;

$$\left.\begin{aligned} \frac{\partial \phi_g}{\partial t} &= L^2_g \nabla^2 \phi_g + \sum_{i=1}^{6} \lambda_i f_{ig} C_i + \sum_{g'=1}^{G} X_{gg'} \phi_{g'} + \theta \sum_{g'=1}^{G} \rho_{gg'} \phi_{g'} + S_g \quad g = 1, .., G \\ \frac{\partial C_i}{\partial t} &= \beta_i \sum_{g'=1}^{G} F_{1g'} \phi_{g'} + \beta_i \phi \sum_{g'=1} F_{2g'} \phi_{g'} - \lambda_i C_i \quad i = 1, .., 6 \\ \frac{\partial \theta}{\partial t} &= m^2 \nabla^2 \theta - \frac{\theta}{t_2} + \sum_{g'=1}^{G} \varepsilon_{1g'} \phi_g + \theta \sum_{g'=1}^{G} \varepsilon_{2g'} \phi_{g'} \end{aligned}\right\} (2)$$

with the boundary conditions $\phi_2(\vec{R}, t) = C_i(\vec{R}, t) = 0$, $\theta(\vec{R}, t) = T_a - T_c$ here T_c is the average coolant temperature and $\theta(\vec{r}, t) = T(\vec{r}, t) - T_c$ and the initial conditions $\phi_g(\vec{r}, 0) = \phi_{g0}(\vec{r})$, $C_i(\vec{r}, 0) = C_{i0}(\vec{r})$, $\theta(\vec{r}, 0) = \theta_0(\vec{r}) = T_0(\vec{r}) - T_c$

where $X_{gg'}$ and $P_{gg'}, F_{1g'}$ and $F_{2g'}$, and $\epsilon_{2g'}$ are proportional to zeroth and first order temperature coefficients for the group fluxes, delayed neutron groups and temperature distribution respectively.

STABILITY THEOREMS THROUGH COMPARISON FUNCTIONS

Eqs. 2 are of the form $\vec{u}_{,t} = \vec{F}(t, \vec{r}, \vec{u}, u_{,rr})$ where the derivative $u_{,rr}$ of the i^{th} component u_i may appear only in the i^{th} component $(\vec{F})_i$. For this set of parabolic differential equations we define $P\vec{u}$ (the defect) as the vector function whose components are $(P\vec{u})_i = u_{i,t} - F_i(t, \vec{r}, \vec{u}, u_{i,rr})$. Note that the defect $P\vec{u}$ for the solution of the system (i.e. $\vec{u}$) is the zero vector.

Let D be an open subset of R^3 (i.e. the reactor volume) D be the closure of D, $G = (0, T) \times D$, ∂D and ∂G are the boundaries of D and G respectively.

$$\partial_0 G = \{0\} \times D,\ \partial_1 G = (0, T) \times \partial D,\ \partial_2 G = \{T\} \times D$$

$$G_p = G + \partial_2 G,\ R_p = \partial_0 G + \partial_1 G$$

Now we state a lemma.

LemmaI: (Nagumo - Westphal Lemma).
Let $\vec{v}$, $\vec{w}$, $\vec{F}$ be suitable given vector functions such that

i) $\vec{v} \leqslant \vec{w}$ on R_p and

ii) $P\vec{v} \leqslant P\vec{w}$ in G_p

Then $\vec{v} \leqslant \vec{w}$ in G_p.

Based on this lemma we obtain the following stability theorem with respect to perturbations in the initial distributions.

Thm: 1- Let $\vec{u}_0 \equiv (\phi_0, \theta_0, C_0)$ be the eguilibrium solution of the system (2) and $\vec{v}(\vec{r}, t)$, $\vec{w}(\vec{r}, t)$ vector functions such that $\vec{v}(\vec{r},0) \leqslant \vec{u}(\vec{r},0) \leqslant \vec{w}(\vec{r},0)$ and $\vec{v}(\vec{R},t) \leqslant \vec{u}(\vec{R},t) \leqslant \vec{w}(\vec{R},t)$

If $P\vec{v} = \vec{v}_{,t} - \vec{F}(t,\vec{r},\vec{v},v_{,rr}) \leqslant P\vec{u} = 0 \leqslant P\vec{w} = \vec{w}_{,t} - \vec{F}(t,\vec{r},\vec{w},w_{,rr})$ $\forall\, t,\vec{r}$ then $\vec{v}(\vec{r},t) \leqslant \vec{u}(\vec{r},t) \leqslant \vec{w}(\vec{r},t)$ $\forall\, t,\vec{r}$

In other words if $\vec{v}$, $\vec{w}$ vector functions can be

chosen so that they envelope the solution of $\vec{u}_{,t} = \vec{F}(t,\vec{r},\vec{u},\vec{u}_{,rr})$ at $t = 0$ (envelope the initial distribution) and at the reactor boundaries for all t, then $\vec{v}$ and $\vec{w}$ envelope the solution for all times throughout the reactor.

In particular $\vec{v}$ and $\vec{w}$ can be chosen as; $\vec{v} = (1-De^{-\gamma t})\vec{u}_0$, $\vec{w} = (1 + Ae^{-\alpha t})\vec{u}_0$ where $0 \leq D < 1$, $A \geq 0$ and $\alpha, \gamma \geq 0$. Clearly $\vec{v}(\vec{r}, 0) = (1-D)\vec{u}_0 \leq u_0 \leq \vec{w}(\vec{r}, 0) = (1+A)\vec{u}_0$ and the same relationship holds at the boundaries also, since $\vec{u}$, $\vec{v}$, $\vec{w}$ satisfy the same boundary conditions. In this case if one can find A, D, γ, α as defined above such that

$$P\vec{v} \leq P\vec{u} = 0 \leq P\vec{w} \text{ i.e. } P\vec{v} \leq 0,\ P\vec{w} \geq 0 \tag{3}$$

then $\vec{v} = (1-De^{-\gamma t})\vec{u}_0 \leq \vec{u}(\vec{r}, t) \leq (1 + Ae^{-\alpha t})\vec{u}_0 = \vec{w}\ \forall t,\ \vec{r}$

Note that $\lim_{t\to\infty} \vec{v} = \lim_{t\to\infty} \vec{w} = \vec{u}_0$. Hence supposing we have α, γ, A, D satisfying Eq.3 then if we perturb the initial equilibrium state $\vec{u}_0$ to $\vec{u}_0'$ at $t = 0$ then the ensuing transient $\vec{u}(\vec{r}, t)$ will be enveloped by $\vec{v}$, $\vec{w}$ at all times and will eventually return to $\vec{u}_0$. This means that $\vec{u}_0$ is asymptotically stable. If α, γ are let to be zero then $\vec{v} = (1 - D)\vec{u}_0$, $\vec{w} = (1 + A)\vec{u}_0$ define a region in which the perturbed solution will be confined. If α, $\gamma > 0$ then they are a measure of the rate of convergence to equilibrium.

The conditions equivalent to Eq.3 for the upper bound w are:

$$-\alpha - \left(\sum_{g'=1}^{G} \rho_{gg',\theta_0} \frac{\phi_{g',0}}{\phi_{g0}}\right)(1 + Ae^{-\alpha t})\frac{\rho_g}{\phi_{g0}} \geq 0 \qquad g = 1,\ldots,G$$

$$-\alpha - \left(\sum_{g'=1}^{G} \varepsilon^2_{g',}\phi_{g',2}\right)(1 + Ae^{-\alpha t}) \geq 0$$

$$-\alpha + \lambda_i - \beta_i \sum_{g'=1}^{G} \{(F_{1g',} + F_{2g',\theta_0}) + F_{2g',\theta_0}(1 + Ae^{-\alpha t})\}\frac{\phi_{g'0}}{C_{i0}} \geq 0 \qquad i = 1,\ldots,6$$

If positive A and α are chosen to satisfy these inequalities then the state $\vec{u}_0$ is asymptotically stable and α gives the exponential rate of convergence to equilibrium. If the inequalities are satisfied with $\alpha = 0$, $A \geq 0$ then it can only be concluded that the system is not unstable.

Similar considerations about perturbations in the system parameters give similar stability results.

Now suppose that the system $\vec{u}_{,t} = \vec{F}(t,\vec{r},\vec{u},u_{,rr})$ has been perturbed while at an equilibrium state $\vec{u}_0$ such that $\vec{F} \to \vec{F}'$ then the following lemma holds for the resulting solution $\vec{u}(\vec{r},t)$.

Lemma II: Suppose $\vec{F}$, $\vec{F}'$ are quasi-monotone increasing and $\vec{q}_1$, $\vec{q}_2$ are vector functions such that

i) $\vec{u}_0 - \vec{q}_2 \leqslant \vec{u} \leqslant \vec{u}_0 + \vec{q}_1$ on Rp

ii) $q_{1,t} \geqslant \vec{F}'(t,\vec{r},\vec{u}_0 + \vec{q}_1, (u_0 + q_1)_{,rr})$

$\vec{q}_{2,t} \geqslant -\vec{F}'(t,\vec{r},\vec{u}_0 - \vec{q}_2, (u_0 - q_2)_{,rr})$ in Gp

then we have $\vec{u}_0 - \vec{q}_2 \leqslant \vec{u} \leqslant \vec{u}_0 + \vec{q}_1$ in Gp

Now suppose in system (2) perturbations are:

$L_g^{'2} \to L_g^{\,2} + \delta L_g^{\,2}$, $X_{gg'} \to X_{gg'} + \delta X_{gg'}$, $\rho_{gg'} \to \rho_{gg'} + \delta\rho_{gg'}$ etc.

Let us choose $\vec{q}_1 = q_1 \vec{u}_0 e^{-\alpha\tau}$ and $\vec{q}_2 = q_2 \vec{u}_0 e^{-\gamma\tau}$ where q_1, q_2, α, $\gamma \geqslant 0$ such that hypothesis (ii) of lemma II is satisfied. Then if α, $\gamma > 0$ then $\vec{u}_0 - \vec{q}_2 \leqslant \vec{u} \leqslant \vec{u}_0 + \vec{q}_1$ implies that $\lim_{t\to\infty} \vec{u}(\vec{r},t) = \vec{u}_0(\vec{r})$ because $\lim_{t\to\infty} \vec{q}_1 = \lim_{t\to\infty} \vec{q}_2 = 0$. Hence the system is asymptotically stable with respect to the perturbations in the parameters that are specified above. However if $\alpha = \gamma = 0$ then the solution ensuing from the perturbations in system parameters is bounded from above and below by $(1 + q_1)\vec{u}_0$ and $(1 - q_2)\vec{u}_0$ respectively.

The inequalities that are to be satisfied by q_1, q_2, α, γ are not reproduced here but they are easily obtained by applying lemma II to the system (2) with the given forms of $\vec{q}_1$ and $\vec{q}_2$

CONCLUSION

In summary, by the comparison function techniques we are able to contstruct functions which envelope the unknown solution of the perturbed system and thus derive conclusions about the stability of the system with respect to different perturbations.

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