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ON THE TWO DIMENSIONAL SOLUTIONS FOR AN ELASTIC SEMI-INFINITE MEDIUM WITH COUPLE STRESS

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E. S. Şuhubi*

P.O. Box 1, Airport, Istanbul, Turkey

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*) Technical University of Istanbul, Istanbul, Turkey

ABSTRACT

The governing differential equations for an elastic medium with couple stress are solved for a semi-infinite region in static plane deformation by using Fourier transforms. The formal solution can be utilized in a large class of boundary value problems concerning the region in question. For an arbitrary distribution of surface tractions and moments explicit results are given and the solution is obtained for the uniformly distributed surface tractions and moments.

1. INTRODUCTION

In a previous work we have derived two coupled vectorial differential equations in field variables for a linear, isotropic elastic solid with couple stress [1]. Using the results obtained there, we now study the plane deformation of a semi-infinite region defined by the equation $y \geq 0$ assuming $u_x = u$, $u_y = v$ and $\phi_z = \phi$ as the only non-vanishing components of the displacement and micro-rotation vectors. We further suppose that the medium is free from body forces and moments. The partial differential equations obtained under these assumptions are reduced to a system of ordinary differential equations by applying Fourier transform in x-direction. These equations are then solved and the transformed components of the displacement and micro-rotation vectors and of the stress and couple stress tensors are obtained. The formulation enables one to deal with a large class of boundary value problems of which loading with arbitrarily distributed surface tractions and moments is considered

here. As a simple example, the solution for the uniformly distributed surface tractions and moments is given.

2. SOLUTION OF THE FUNDAMENTAL EQUATIONS

The governing vectorial differential equations [1] can be written for a plane deformation characterized by the field

$$u_x = u(x,y) \ , \quad u_y = v(x,y) \ , \quad u_z = 0 \ ,$$
$$\phi_x = 0 \ , \quad \phi_y = 0 \ , \quad \phi_z = \phi(x,y) \tag{2.1}$$

as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\,\partial y} + \frac{\mu+\kappa}{\lambda+\mu-\kappa}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\kappa}{\lambda+\mu-\kappa}\frac{\partial \phi}{\partial y} = 0 \ ,$$

$$\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} + \frac{\mu+\kappa}{\lambda+\mu-\kappa}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{\kappa}{\lambda+\mu-\kappa}\frac{\partial \phi}{\partial x} = 0 \ , \tag{2.2}$$

$$\frac{\alpha+\beta-\delta}{2(\lambda+\mu-\kappa)}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) - \frac{\kappa}{\lambda+\mu-\kappa}\left(\phi + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0 \ .$$

If we define Fourier transform $\bar{f}(\xi,y)$ of a function $f(x,y)$ by

$$\bar{f}(\xi,y) = \int_{-\infty}^{\infty} f(x,y)\, e^{i\xi x}\, dx$$

the foregoing equations are reduced to the following system of ordinary differential equations

$$\frac{\mu+\kappa}{\lambda+\mu-\kappa}\bar{u}'' - \frac{\lambda+2\mu}{\lambda+\mu-\kappa}\xi^2\bar{u} - i\xi\bar{v}' - \frac{\kappa}{\lambda+\mu-\kappa}\bar{\phi}' = 0 \ ,$$

$$-i\xi\bar{u}' + \frac{\lambda+2\mu}{\lambda+\mu-\kappa}\bar{v}'' - \frac{\mu+\kappa}{\lambda+\mu-\kappa}\bar{v} - i\frac{\kappa}{\lambda+\mu-\kappa}\xi\bar{\phi} = 0 \ ,$$

$$- \frac{\kappa}{\lambda+\mu-\kappa} \bar{u}' - i\frac{\kappa}{\lambda+\mu-\kappa} \xi \bar{v} - \frac{\alpha+\beta-\delta}{2(\lambda+\mu-\kappa)} \bar{\phi}''$$

$$+ \left(\frac{\kappa}{\lambda+\mu-\kappa} + \frac{\alpha+\beta-\delta}{2(\lambda+\mu-\kappa)} \xi^2 \right) \bar{\phi} = 0 \qquad (2.3)$$

where primes denote differentiations with respect to y.

The characteristic equation for this system is

$$(D^2 - \xi^2)^2[D^2 - \xi^2 - \Lambda^2] = 0 \qquad (2.4)$$

where

$$\Lambda = \sqrt{\frac{2\kappa\mu}{(\mu+\kappa)\ (\alpha+\beta-\delta)}} \quad , \ [\Lambda] = L^{-1} \qquad (2.5)$$

which is a real quantity since κ and $\alpha+\beta-\delta$ are positive [2]. A typical solution can thus be written as

$$f(\xi,y) = Ae^{-|\xi|y} + Bye^{-|\xi|y} + Ce^{-\sqrt{\xi^2+\Lambda^2}y} + De^{|\xi|y} + Fye^{|\xi|y} + Ge^{\sqrt{\xi^2+\Lambda^2}y}$$

A, . . . , G being arbitrary functions of ξ .

In a semi-infinite medium in order that the field quantities remain bounded at the infinity we have to take

$$D = F = G = 0 \ .$$

In this case the solution of the system (2.3) is found to be

$$\bar{u}(\xi,y) = Ae^{-|\xi|y} + Bye^{-|\xi|y} + \sqrt{\xi^2+\Lambda^2}\, Ce^{-\sqrt{\xi^2+\Lambda^2}y} ,$$

$$\bar{v}(\xi,y) = -i\{[\operatorname{sgn}\xi A + \frac{\lambda+3\mu}{(\lambda+\mu)\xi} B]e^{-|\xi|y} + \operatorname{sgn}\xi Bye^{-|\xi|y} + \xi Ce^{-\sqrt{\xi^2+\Lambda^2}y} , \tag{2.6}$$

$$\bar{\phi}(\xi,y) = 2[\frac{\lambda+2\mu}{\lambda+\mu} Be^{-|\xi|y} - \frac{\mu}{\alpha+\beta-\delta} Ce^{-\sqrt{\xi^2+\Lambda^2}y}] .$$

Having the expressions for $\bar{u}$, $\bar{v}$ and $\bar{\phi}$, the transformed components of the stress and couple stress tensors are easily obtained. If we recall that $\phi_\kappa = e_{\kappa\ell m}\phi_{\ell m}$, the non-vanishing components of these tenors can be written as follows [2]:

$$t_{11} = \lambda(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + 2\mu\frac{\partial u}{\partial x} ,$$

$$t_{22} = \lambda(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + 2\mu\frac{\partial v}{\partial y} ,$$

$$t_{33} = \lambda(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = \frac{\lambda}{2(\lambda+\mu)}(t_{11} + t_{22}) , \tag{2.7}$$

$$t_{12} = (\mu - \kappa)\frac{\partial u}{\partial y} + (\mu + \kappa)\frac{\partial v}{\partial x} + \kappa\phi ,$$

$$t_{21} = (\mu + \kappa)\frac{\partial u}{\partial y} + (\mu - \kappa)\frac{\partial v}{\partial x} - \kappa\phi ,$$

$$m_{112} = -\frac{\alpha+\beta-\delta}{2}\frac{\partial\phi}{\partial x} ,$$

$$m_{212} = -\frac{\alpha+\beta-\delta}{2}\frac{\partial\phi}{\partial y} ,$$

$$m_{313} = -\frac{\alpha}{2}\frac{\partial\phi}{\partial y} ,$$

$$m_{323} = \frac{\alpha}{2}\frac{\partial\phi}{\partial x} .$$

Therefore their transforms take the following forms:

$$
\begin{aligned}
\bar{t}_{11} &= -i(\lambda+2\mu)\xi\bar{u} + \lambda\bar{v}' \quad , \\
\bar{t}_{22} &= -i\lambda\xi\bar{u} + (\lambda+2\mu)\bar{v}' \quad , \\
\bar{t}_{33} &= \frac{\lambda}{2(\lambda+\mu)}(\bar{t}_{11} + \bar{t}_{22}) \quad , \\
\bar{t}_{12} &= (\mu-\kappa)\bar{u}' - i(\mu+\kappa)\xi\bar{v} + \kappa\bar{\phi}, \\
\bar{t}_{21} &= (\mu+\kappa)\bar{u}' - i(\mu-\kappa)\xi\bar{v} - \kappa\bar{\phi} \quad , \\
2\bar{m}_{112} &= i(\alpha+\beta-\delta)\xi\bar{\phi} \quad , \\
2\bar{m}_{212} &= -(\alpha+\beta-\delta)\bar{\phi}' \quad , \\
2\bar{m}_{313} &= -\alpha\bar{\phi}' \quad , \\
2\bar{m}_{323} &= -i\alpha\xi\bar{\phi} \quad .
\end{aligned}
\tag{2.8}
$$

Introducing (2.6) into (2.8) we obtain for the transformed components the following expressions which involve the functions A, B and C that are yet unknown:

$$
\begin{aligned}
\bar{t}_{11} &= -2i\mu[(\xi A - \frac{\lambda}{\lambda+\mu}\,\mathrm{sgn}\xi B)e^{-|\xi|y} + \xi B y e^{-|\xi| y} + \xi\sqrt{\xi^2+\Lambda^2}\,Ce^{-\sqrt{\xi^2+\Lambda^2}\,y}\,], \\
\bar{t}_{22} &= 2i\mu[(\xi A + \frac{\lambda+2\mu}{\lambda+\mu}\,\mathrm{sgn}\xi B)e^{-|\xi|y} + \xi B y e^{-|\xi|y} + \xi\sqrt{\xi^2+\Lambda^2}\,Ce^{-\sqrt{\xi^2+\Lambda^2}\,y}\,] \quad , \\
\bar{t}_{33} &= 2i\,\frac{\lambda\mu}{\lambda+\mu}\,\mathrm{sgn}\xi\, Be^{-|\xi|y} \quad , \\
\bar{t}_{12} &= -2\mu[(|\xi|A + \frac{\mu}{\lambda+\mu}B)e^{-|\xi|y} + |\xi| B y e^{-|\xi|y} + (\xi^2+\Lambda^2)Ce^{-\sqrt{\xi^2+\Lambda^2}\,y}\,] \quad , \\
\bar{t}_{21} &= -2\mu[(|\xi|A + \frac{\mu}{\lambda+\mu}B)e^{-|\xi|y} + |\xi| B y e^{-|\xi|y} + \xi^2 Ce^{-\sqrt{\xi^2+\Lambda^2}\,y}\,] \quad , \\
\bar{t}_{21} - \bar{t}_{12} &= 2\mu\Lambda^2 Ce^{-\sqrt{\xi^2+\Lambda^2}\,y} \quad , \\
\bar{m}_{112} &= i(\alpha+\beta-\delta)\xi[\,\frac{\lambda+2\mu}{\lambda+\mu}Be^{-|\xi|y} - \frac{\mu}{\alpha+\beta-\delta}Ce^{-\sqrt{\xi^2+\Lambda^2}\,y}\,] \quad ,
\end{aligned}
$$

$$\bar{m}_{212} = (\alpha+\beta-\delta)\left[\frac{\lambda+2\mu}{\lambda+\mu}|\xi| B e^{-|\xi|y} - \frac{\mu}{\alpha+\beta-\delta}\sqrt{\xi^2+\Lambda^2}\, C e^{-\sqrt{\xi^2+\Lambda^2}y}\right],$$

$$\bar{m}_{313} = \frac{\alpha}{\alpha+\beta-\delta}\,\bar{m}_{212},$$

$$\bar{m}_{323} = -\frac{\alpha}{\alpha+\beta-\delta}\,\bar{m}_{112}. \qquad (2.9)$$

Specifying any three of the u, v, ϕ, t_{22}, t_{21} and m_{212} on the boundary y=0 we are furnished with three linear equations which enables us to determine A(ξ), B(ξ) and C(ξ). On determining these functions we can obtain the original values of the desired functions with the aid of inverse transform

$$f(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}(\xi,y)\, e^{-i\xi x}\, d\xi$$

3. *ARBITRARILY DISTRIBUTED SURFACE TRACTIONS AND MOMENTS ON THE BOUNDARY*

We now assume that the normal and tangential tractions and surface moments in z-direction are prescribed in an arbitrary way on the boundary. Boundary conditions take, in this case, the following forms [3, p. 102]:

$$t_{22}(x,0) = \sigma(x), \quad t_{21}(x,0) = \tau(x), \quad 2m_{212}(x,0) = m(x)$$

or in transformed field variables

$$\bar{t}_{22}(\xi,0) = \bar{\sigma}(\xi), \quad \bar{t}_{21}(\xi,0) = \bar{\tau}(\xi), \quad 2\bar{m}_{212}(\xi,0) = \bar{m}(\xi). \qquad (3.1)$$

Putting y=0 in $(2.9)_2$, $(2.9)_5$ and $(2.9)_8$ and equating the resulting expressions to $(3.1)_1$, $(3.1)_2$ and $(3.1)_3$ respectively, we get a system of linear equations in A, B, and C whose solution is given below:

$$A = \frac{1}{2\mu\xi^2\Delta(\xi)} \{i\xi[\mu(\mu+\kappa)\Lambda^2\sqrt{\xi^2+\Lambda^2} + 2\kappa(\lambda+2\mu)|\xi|\xi^2]\bar{\sigma}(\xi)$$

$$- (\lambda+2\mu)|\xi|[(\mu+\kappa)\Lambda^2 + 2\kappa\xi^2]\sqrt{\xi^2+\Lambda^2}\ \bar{\tau}(\xi)$$

$$+ (\mu+\kappa)\Lambda^2\xi^2[(\lambda+2\mu)|\xi| - \mu\sqrt{\xi^2+\Lambda^2}]\ \bar{m}(\xi)$$

$$B = \frac{(\lambda+\mu)(\mu+\kappa)\Lambda^2}{2\mu|\xi|\Delta(\xi)} [-i\xi\sqrt{\xi^2+\Lambda^2}\bar{\sigma}(\xi) + |\xi|\sqrt{\xi^2+\Lambda^2}\bar{\tau}(\xi)$$

$$+ \xi^2(\sqrt{\xi^2+\Lambda^2} - |\xi|)\bar{m}(\xi)]$$

$$C = \frac{1}{2\mu\Delta(\xi)} [-2i\kappa(\lambda+2\mu)\xi\bar{\sigma}(\xi) + 2\kappa(\lambda+2\mu)|\xi|\bar{\tau}(\xi)$$

$$- (\mu+\kappa)(\lambda+\mu)\Lambda^2\bar{m}(\xi)\ ] \qquad (3.2)$$

where

$$\Delta(\xi) = [(\lambda+\mu)(\mu+\kappa)\Lambda^2 + 2\kappa(\lambda+2\mu)\xi^2]\sqrt{\xi^2+\Lambda^2} - 2\kappa(\lambda+2\mu)|\xi|\xi^2. \qquad (3.3)$$

In the classical limit κ, α, β, δ must obviously vanish. Furthermore, if we want that the vector $\underset{\sim}{\phi}$ represent the classical rotation of an infinitesimal element we have also to assume that, in the classical limit, [see Eq. $(2.2)_3$]

$$\frac{\alpha+\beta-\delta}{2\kappa} \to 0 \quad \text{or} \quad \Lambda \to \infty \ .$$

If we let Λ approach to infinity in the expressions (3.2) we can indeed find the classical results [4]:

$$A = i\frac{\bar{\sigma}(\xi)}{2(\lambda+\mu)\xi} - \frac{(\lambda+2\mu)\bar{\tau}(\xi)}{2\mu(\lambda+\mu)|\xi|} ,$$

$$B = -i\frac{\operatorname{sgn}\xi\bar{\sigma}(\xi)}{2\mu} + \frac{\bar{\tau}(\xi)}{2\mu} , \qquad C = 0 .$$

Let us finally consider as a very simple example, the uniformly distributed normal and tangential tractions and moments in z-direction on the boundary. Their magnitudes per unit length are denoted by σ_0, τ_0, and m_0 respectively. The transformed boundary values are

$$\bar{\sigma}(\xi) = 2\pi\sigma_0\delta(\xi) , \quad \bar{\tau}(\xi) = 2\pi\tau_0\delta(\xi) , \quad \bar{m}(\xi) = \pi m_0\delta(\xi) \qquad (3.4)$$

where $\delta(\xi)$ is Dirac's delta function. Introducing (3.4) into (3.2) and (3.2) into any desired function in (2.9) and taking the inverse transform one obtains

$$\begin{aligned}
t_{11} &= \sigma_0 , & 2m_{212} &= m_{23} = m_0 e^{-\Lambda y} , \\
t_{22} &= \sigma_0 , & 2m_{331} &= m_{32} = -\frac{\alpha}{\alpha+\beta-\delta} m_0 e^{-\Lambda y} . \\
t_{33} &= -\frac{\lambda}{\lambda+\mu}\sigma_0 , & & \\
t_{12} &= \tau_0 + m_0\Lambda e^{-\Lambda y} , & & \\
t_{21} &= \tau_0 , & &
\end{aligned} \qquad (3.5)$$

It is easily seen that the effect of normal and tangential tractions at the boundary is the same as the one in the classical theory. Only the surface moments give rise to effects which have no counterpart in the classical theory. In common materials one reasonably expects

that Λ takes very large values so that these effects will be confined in a very thin layer on the surface of the medium.

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